



# A two-sample test for comparison of long memory parameters

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## ARTICLE INFO

### Article history:

Received 10 July 2009

Available online 24 April 2010

### AMS subject classifications:

60F05

62M10

60G22

### Keywords:

Long memory

Two-sample test for comparison of memory parameters

Long-run covariance

Rescaled variance

bandwidth choice

## ABSTRACT

We construct a two-sample test for comparison of long memory parameters based on ratios of two rescaled variance (V/S) statistics studied in Giraitis et al. [L. Giraitis, R. Leipus, A. Philippe, A test for stationarity versus trends and unit roots for a wide class of dependent errors, *Econometric Theory* 21 (2006) 989–1029]. The two samples have the same length and can be mutually independent or dependent. In the latter case, the test statistic is modified to make it asymptotically free of the long-run correlation coefficient between the samples. To diminish the sensitivity of the test on the choice of the bandwidth parameter, an adaptive formula for the bandwidth parameter is derived using the asymptotic expansion in Abadir et al. [K. Abadir, W. Distaso, L. Giraitis, Two estimators of the long-run variance: beyond short memory, *Journal of Econometrics* 150 (2009) 56–70]. A simulation study shows that the above choice of bandwidth leads to a good size of our comparison test for most values of fractional and ARMA parameters of the simulated series.

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## 1. Introduction

Long memory is one of the most widely discussed “stylized facts” of financial time series (see, e.g. [24]). In real data, long memory can be confused with short memory, unit roots, trends, structural changes, heavy tails and other features. Various tests for long memory have been developed in the literature. See [17,15,19,18,11,12,23]. Most of these results pertain to the case of a single sample.

A natural extension of one-sample test about unknown long memory parameter  $d$  is two-sample testing for comparison of respective memory parameters  $d_1$  and  $d_2$ . In particular, such a test can be useful for the memory propagation (from durations to counts and realised volatility), question discussed in [9]. Several studies compare the estimates of long memory parameter from different foreign exchange data and other sources [5,21,4]. Two-sample testing is also related to the change-point problem of the memory parameter discussed in [2,13].

The present paper constructs a test for testing the null hypothesis  $d_1 = d_2$  that long memory parameters  $d_i \in [0, 1/2)$  of two samples of length  $n$ , taken from respective stationary processes  $X_i$ ,  $i = 1, 2$ , are equal, against the alternative  $d_1 \neq d_2$ . The test statistic,  $T_n$ , is defined as a sum

$$T_n = \frac{V_1/S_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{V_1/S_{11,q}}, \quad (1.1)$$

of two ratios of V/S, or rescaled variance, statistics  $V_1/S_{11,q}$  and  $V_2/S_{22,q}$  computed from samples  $(X_1(1), \dots, X_1(n))$  and  $(X_2(1), \dots, X_2(n))$ . Here,  $V_i$  is the empirical variance of partial sums of  $X_i$  and  $S_{ii,q}$  is the Newey–West or HAC estimator

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of the long-run variance of  $X_i$ . The  $V/S$  statistic was developed in [11,12] following the works of Lo [17] and Kwiatkowski et al. [15] on related  $R/S$  type statistics. In particular, from [12] one easily derives the asymptotic null distribution  $T$  of the statistic  $T_n$  under the condition that the two samples are independent. It is also easy to show that for  $d_1 \neq d_2$ , one of the ratios in (1.1) tends to infinity and the other one to zero, meaning that the test is consistent against the alternative  $d_1 \neq d_2$ .

However, independence of the two samples is too restrictive and may be unrealistic in financial data analysis since price movements of different assets are usually correlated and susceptible to common macroeconomic shocks. In order to eliminate the eventual dependence between samples, a modification  $\tilde{T}_n$  of (1.1) is proposed, which uses residual observations  $(\tilde{X}_1(1), \dots, \tilde{X}_1(n))$ , obtained by regressing partial sums of  $X_1$  on partial sums of  $X_2$ . The modified statistic  $\tilde{T}_n$  is shown to have the same limit null distribution  $T$  as if the two samples were independent.

It is well-known that a major problem in applications of the rescaled variance and related statistics is the choice of the bandwidth parameter  $q$ . The present paper contributes to this problem by providing an adaptive formula in (A.32) for “optimal”  $q$  which depends not only on the (common) memory parameter  $d$  but also on the difference between estimated short memory (AR) components of the spectrum of the sampled series. The derivation of the last result uses the expansion of the HAC estimator in [1]. A simulation study confirms that using this choice of bandwidth leads to a good size of our comparison tests for most values of fractional and ARMA parameters of the simulated series.

Several interesting open problems were suggested by Referee and Associated Editor. The assumption of stationarity can be very restrictive for applications. We expect that our results can be extended to values of  $d_i$  outside the interval  $[0, 1/2)$ . Another possibility for future research is development of similar procedures to test equal memory for more than two series.

The plan of the paper is as follows. Section 2 formulates the settings of the paper (Assumptions A( $d_1, d_2$ ) and B( $d_1, d_2$ )) and derives the limit of the test statistics  $T_n$  and  $\tilde{T}_n$  and the rejection region of the null hypothesis  $d_1 = d_2$ . Assumption A( $d_1, d_2$ ) guarantees the existence of long-run (co)variances and the consistency of the HAC estimators. Assumption B( $d_1, d_2$ ) specifies the joint limit behavior of partial sums of  $X_1$  and  $X_2$  as given by bivariate fractional Brownian motion. The last process is defined by means of stochastic integral representation as in [6]. The test procedures are then presented in detail and a brief study focus on the asymptotic power of the tests  $T_n$  and  $\tilde{T}_n$ . Section 3 verifies Assumptions A( $d_1, d_2$ ) and B( $d_1, d_2$ ) for bivariate moving average ( $X_1, X_2$ ). Section 4 assesses the performance of the tests  $T_n$  and  $\tilde{T}_n$ , by simulating bivariate FARIMA samples with various fractional and autoregressive/moving average parameters. Conclusions are given in Section 5. The Appendix contains auxiliary results and derivations.

**Notation.** Below,  $\rightarrow_p$ ,  $\rightarrow_{\text{law}}$ ,  $\rightarrow_{D[0,1]}$  and  $\rightarrow_{\text{fdd}}$  ( $=\text{fdd}$ ) stand for the convergence in probability, the weak convergence of random variables, the weak convergence of random elements in the Skorohod space  $D[0, 1]$ , and the weak convergence (equality) of finite dimensional distributions, respectively. Relation ‘ $\sim$ ’ means that the ratio of both sides tends to 1.

## 2. Construction of tests and its properties

### 2.1. Assumptions and main results

Let  $((X_1(t), X_2(t)), t \in \mathbb{Z})$  be a bivariate covariance stationary process, viz., a sequence of random vectors  $(X_1(t), X_2(t)) \in \mathbb{R}^2$  such that  $\text{EX}_i(t) = \mu_i$  and

$$\text{cov}(X_i(t), X_j(t+h)) = \gamma_{ij}(h)$$

do not depend on  $t \in \mathbb{Z}$  for any  $h \in \mathbb{Z}$ ,  $i, j = 1, 2$ . Introduce the popular Bartlett-kernel estimator of the long-run (co)variance:

$$S_{ij,q} = \sum_{h=-q}^q \left(1 - \frac{|h|}{q+1}\right) \hat{\gamma}_{ij}(h), \quad (2.1)$$

where

$$\hat{\gamma}_{ij}(h) = n^{-1} \begin{cases} \sum_{t=1}^{n-h} (X_i(t) - \bar{X}_i)(X_j(t+h) - \bar{X}_j), & h \geq 0, \\ \sum_{t=1-h}^n (X_i(t) - \bar{X}_i)(X_j(t+h) - \bar{X}_j), & h \leq 0, \end{cases} \quad (2.2)$$

$\bar{X}_i = n^{-1} \sum_{t=1}^n X_i(t)$ . The estimator in (2.1) is also called the heteroskedasticity and autocorrelation consistent (HAC). Also define

$$S_{ij,q}^\circ = \sum_{h=-q}^q \left(1 - \frac{|h|}{q+1}\right) \hat{\gamma}_{ij}^\circ(h), \quad (2.3)$$

where

$$\hat{\gamma}_{ij}^{\circ}(h) = n^{-1} \begin{cases} \sum_{t=1}^{n-h} (X_i(t) - \mu_i)(X_j(t+h) - \mu_j), & h \geq 0, \\ \sum_{t=1-h}^n (X_i(t) - \mu_i)(X_j(t+h) - \mu_j), & h \leq 0. \end{cases} \quad (2.4)$$

Note  $\hat{\gamma}_{ij}^{\circ}(h) = \hat{\gamma}_{ji}^{\circ}(-h)$ ,  $\hat{\gamma}_{ij}^{\circ}(h) = \hat{\gamma}_{ji}^{\circ}(-h)$ ,  $S_{12,q} = S_{21,q}$ ,  $S_{12,q}^{\circ} = S_{21,q}^{\circ}$ .

**Assumption A**( $d_1, d_2$ ). There exist  $d_i \in [0, 1/2)$ ,  $i = 1, 2$  such that for any  $i, j = 1, 2$  the following limits exist

$$\begin{aligned} c_{ij} &= \lim_{n \rightarrow \infty} \frac{1}{n^{1+d_i+d_j}} \mathbb{E} \left( \sum_{t=1}^n (X_i(t) - \mu_i) \right) \left( \sum_{s=1}^n (X_j(s) - \mu_j) \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^{1+d_i+d_j}} \sum_{t,s=1}^n \gamma_{ij}(t-s). \end{aligned} \quad (2.5)$$

Moreover,

$$\frac{\sum_{k,l=1}^q \hat{\gamma}_{ij}(k-l)}{\sum_{k,l=1}^q \gamma_{ij}(k-l)} \rightarrow_p 1 \quad (2.6)$$

as  $q \rightarrow \infty$ ,  $n \rightarrow \infty$ ,  $n/q \rightarrow \infty$ .

**Remark 2.1.** The asymptotic constant  $c_{ij}$  is called the long-run covariance of  $X_i$  and  $X_j$ . Condition (2.6) is similar to [12, Assumption A.2]. It guarantees the consistency of the HAC estimator (see below).

**Proposition 2.2.** Let Assumption A( $d_1, d_2$ ) hold. Then, as  $q \rightarrow \infty$ ,  $n \rightarrow \infty$ ,  $n/q \rightarrow \infty$ ,

$$q^{-d_i-d_j} S_{ij,q} \rightarrow_p c_{ij}, \quad q^{-d_i-d_j} S_{ij,q}^{\circ} \rightarrow_p c_{ij} \quad (i, j = 1, 2). \quad (2.7)$$

Moreover,

$$\frac{1}{q} (S_{ij,q} - S_{ij,q}^{\circ}) = -(\bar{X}_i - \mu_i)(\bar{X}_j - \mu_j) (1 + o_p(1)). \quad (2.8)$$

**Assumption B**( $d_1, d_2$ ), below, specifies the joint limit of partial sums of  $X_1$  and  $X_2$ . It is similar to [12, Assumption A.1]. The limit process (bivariate fractional Brownian motion) is defined through a stochastic integral representation (2.9) similarly as in [6, (6)]. Equivalently, this process can be defined through the covariance function defined in (A.1).

**Definition 2.3.** A nonanticipative bivariate fractional Brownian motion (bi-fBm) with memory parameters  $d_i \in (-1/2, 1/2)$ ,  $i = 1, 2$ , is a Gaussian process  $B = ((B_1(s), B_2(s)), s \in \mathbb{R})$  admitting the following representation for  $i = 1, 2$

$$B_i(t) = \begin{cases} c(d_i) \int_{\mathbb{R}} \left( (t-x)_+^{d_i} - (-x)_+^{d_i} \right) W_i(dx), & \text{if } d_i \neq 0, \\ W_i(0, t], & \text{if } d_i = 0, \end{cases} \quad (2.9)$$

$W = ((W_1(dx), W_2(dx)), x \in \mathbb{R})$  is a 2-dimensional Gaussian independently scattered white noise with real components, zero mean and covariance matrix

$$EW_i(dx)W_j(dx) = dx \begin{cases} 1, & i = j, \\ \rho_W, & i \neq j, \end{cases} \quad (2.10)$$

for some  $\rho_W \in [-1, 1]$ , and the constants  $c(d_i)$  are determined by condition  $EB_i^2(1) = 1$  so that

$$c^2(d_i) = \left( \int \left( (1-x)_+^{d_i} - (-x)_+^{d_i} \right)^2 dx \right)^{-1} = \frac{\cos(d_i\pi)}{B(d_i+1, d_i+1)},$$

where  $B(p, q)$  is the beta function and  $x_+ = \max(x, 0)$ .

**Remark 2.4.** When  $d_1 d_2 \neq 0$ , the nonanticipative bi-fBm is a particular case of general bi-fBm having the stochastic representation

$$X(t) = \int_{\mathbb{R}} \{((t-x)_+^D - (-x)_+^D) A_+ + ((t-x)_-^D - (-x)_-^D) A_-\} \tilde{W}(dx),$$

where  $D = \text{diag}(d_1, d_2)$ ,  $x_- = \max(-x, 0)$ ,  $A_+$ ,  $A_-$  are real  $2 \times 2$  matrices and  $\tilde{W}(dx) = (\tilde{W}_1(dx), \tilde{W}_2(dx))$ ,  $x \in \mathbb{R}$ , is an independently scattered Gaussian white noise with zero mean, unit variance and independent components; see [10], also [16, (1.6)]. The representation in (2.9) corresponds to the matrices  $A_+ A_+^* = \begin{pmatrix} c(d_1)^2 & c(d_1)c(d_2)\rho \\ c(d_1)c(d_2)\rho & c(d_2)^2 \end{pmatrix}$ ,  $A_- = 0$ .

**Remark 2.5.** In the sequel by bi-fBm we mean the nonanticipative process in (2.9). A bi-fBm has stationary increments and the self-similarity property:

$$(\lambda^{-d_1-0.5} B_1(\lambda t), \lambda^{-d_2-0.5} B_2(\lambda t)) =_{\text{fdd}} (B_1(t), B_2(t))$$

for any  $\lambda > 0$ . Lavancier et al. [16] showed that these properties essentially determine the covariance function in (A.1)–(A.3), up to a choice of constants  $g_{ij}$  and  $g_i$  defined in (A.2) and (A.4) (see Appendix A.1).

Note that each component  $B_i$  is a univariate fractional Brownian motion with variance  $EB_i^2(t) = |t|^{2d_i+1}$ ,  $i = 1, 2$ . Also note that  $(B_1(1), B_2(1))$  has a bivariate Gaussian distribution with zero means, unit variances and the correlation coefficient  $\rho = EB_1(1)B_2(1) = \rho_W \kappa(d_1, d_2)$ , where  $\kappa(d_1, d_2)$  depends only on  $d_1, d_2$ . In the case  $d_1 = d_2$ , we have that  $\rho = \rho_W$  and the process  $(\tilde{B}_1(t) = B_1(t) - \rho B_2(t), t \in \mathbb{R})$  is a fractional Brownian motion with variance  $E\tilde{B}_1^2(t) = (1 - \rho^2)|t|^{2d_1+1}$ . Moreover, the processes  $\tilde{B}_1$  and  $B_2$  are independent. Indeed, from (2.9) it is immediate that  $\tilde{B}_1$  has a similar stochastic representation with  $W_1(dx)$  replaced by  $\tilde{W}_1(dx) = W_1(dx) - \rho W_2(dx)$ , with  $\tilde{W}_1$  independent of  $W_2$ .

**Assumption B( $d_1, d_2$ ).** Assumption A( $d_1, d_2$ ) is satisfied and, moreover,

$$\left( n^{-d_1-(1/2)} \sum_{t=1}^{[n\tau]} (X_1(t) - EX_1(t)), n^{-d_2-(1/2)} \sum_{t=1}^{[n\tau]} (X_2(t) - EX_2(t)) \right) \rightarrow_{\text{fdd}} (\sqrt{c_{11}}B_1(\tau), \sqrt{c_{22}}B_2(\tau)), \quad (2.11)$$

where  $c_{ij}$  are the same as in (2.5),  $c_{ii} > 0$ ,  $i = 1, 2$  and  $(B_1, B_2)$  is a bi-fBm with memory parameters  $d_1, d_2$  and the correlation coefficient  $\rho = \text{corr}(B_1(1), B_2(1)) = c_{12}/\sqrt{c_{11}c_{22}} \in [-1, 1]$ .

Define the empirical variance of partial sums of  $X_i$ :

$$V_i = n^{-2} \sum_{k=1}^n \left( \sum_{t=1}^k (X_i(t) - \bar{X}_i) \right)^2 - n^{-3} \left( \sum_{k=1}^n \sum_{t=1}^k (X_i(t) - \bar{X}_i) \right)^2. \quad (2.12)$$

The following proposition obtains the limit distribution of the statistic  $T_n$  defined in (1.1).

**Proposition 2.6.** Let Assumptions A( $d_1, d_2$ ) and B( $d_1, d_2$ ) be satisfied with some  $d_1, d_2 \in [0, 1/2)$  and  $\rho \in [-1, 1]$ , and let  $n, q, n/q \rightarrow \infty$ .

(i) If  $d_1 = d_2 = d$  then

$$T_n \rightarrow_{\text{law}} T = \frac{U_1}{U_2} + \frac{U_2}{U_1}, \quad (2.13)$$

where

$$U_i = \int_0^1 (B_i^0(\tau))^2 d\tau - \left( \int_0^1 B_i^0(\tau) d\tau \right)^2, \quad i = 1, 2, \quad (2.14)$$

where  $(B_i^0(\tau) = B_i(\tau) - \tau B_i(1), \tau \in [0, 1])$ ,  $i = 1, 2$  are fractional Brownian bridges obtained from bivariate fBm  $((B_1(\tau), B_2(\tau)), \tau \in \mathbb{R})$  with the same memory parameters  $d_1 = d_2 = d$  and correlation coefficient  $\rho = \rho_W$  (see Definition 2.3).

(ii) If  $d_1 \neq d_2$  then

$$T_n \rightarrow_p \infty. \quad (2.15)$$

Let  $\tilde{V}_1, \tilde{S}_{11,q}$  be the statistics in (2.12), (2.1), respectively, where  $X_1(t)$ ,  $t = 1, \dots, n$  is replaced by

$$\tilde{X}_1(t) = X_1(t) - (S_{12,q}/S_{22,q})X_2(t), \quad t = 1, \dots, n. \quad (2.16)$$

In particular, note

$$\tilde{S}_{11,q} = S_{11,q} - \frac{S_{12,q}^2}{S_{22,q}}. \quad (2.17)$$

Define

$$\tilde{T}_n = \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{\tilde{V}_1/\tilde{S}_{11,q}}. \quad (2.18)$$

Note,  $\tilde{T}_n$  is obtained by replacing  $V_1, S_{11,q}$  in the definition of  $T_n$  in (2.1) by the corresponding quantities  $\tilde{V}_1, \tilde{S}_{11,q}$  as defined above.

In the following proposition, we prove that under the null hypothesis, the limit distribution of  $\tilde{T}_n$  is free of  $\rho$ , contrary to  $T$  in (2.14). Note that the limit of  $\tilde{T}_n$  coincides with (2.14) when  $\rho = 0$ . This occurs for example when the statistics  $T_n$  is calculated from two independent processes  $X_1, X_2$ .

**Proposition 2.7.** Let Assumptions A( $d_1, d_2$ ) and B( $d_1, d_2$ ) be satisfied with some  $d_1, d_2 \in [0, 1/2)$  and  $\rho \in (-1, 1)$ , and let  $n, q, n/q \rightarrow \infty$ .

(i) If  $d_1 = d_2 = d$  then

$$\tilde{T}_n \rightarrow_{\text{law}} \tilde{T} = \frac{\hat{U}_1}{U_2} + \frac{U_2}{\hat{U}_1}, \quad (2.19)$$

where  $\hat{U}_1, U_2$  are independent and have the same distribution in (2.14).

(ii) If  $d_1 > d_2$  then

$$\tilde{T}_n \rightarrow_p \infty. \quad (2.20)$$

(iii) If  $d_1 < d_2$  then

$$\tilde{T}_n \rightarrow_p \frac{\rho^2}{1 - \rho^2} + \frac{1 - \rho^2}{\rho^2}. \quad (2.21)$$

**Remark 2.8.** The ratio  $\hat{\beta} = S_{12,q}/S_{22,q}$  in (2.16) minimizes the sum of squares:

$$\sum_{i=1-q}^n \left( \sum_{t=i \vee 1}^{(i+q) \wedge n} (X_1(t) - \bar{X}_1) - \hat{\beta} \sum_{t=i \vee 1}^{(i+q) \wedge n} (X_2(t) - \bar{X}_2) \right)^2.$$

Therefore,  $\hat{\rho} = \hat{\beta} \sqrt{S_{22,q}/S_{11,q}}$  can be considered as the least squares estimate of the long-run correlation coefficient  $\rho$  between partial sums of the two samples.

## 2.2. Testing procedures

Let  $t_\alpha(d)$  denote the upper  $\alpha$ -quantile of the r.v.  $\tilde{T}$  defined in (2.19) (or  $T$  in (2.14) when  $\rho = 0$ ), viz.

$$\alpha = P(\tilde{T} > t_\alpha(d)), \quad \alpha \in (0, 1). \quad (2.22)$$

Let

$$\hat{d} = (\hat{d}_1 + \hat{d}_2)/2, \quad (2.23)$$

where  $\hat{d}_i$  is an estimator of  $d_i$  satisfying

$$\hat{d}_i - d_i = o_p(1/\log n) \quad (i = 1, 2). \quad (2.24)$$

Similarly to [12, Lemma 2.1], it can be proved that the quantile function  $t_\alpha(d)$  is continuous in  $d \in [0, 1/2)$  for any  $\alpha \in (0, 1)$ . Therefore, the estimated quantile  $t_\alpha(\hat{d}) \rightarrow_p t_\alpha(d)$  as  $n \rightarrow \infty$  and the asymptotic level of the tests associated to the critical regions in (2.25)–(2.26) is preserved by replacing  $t_\alpha(d)$  by  $t_\alpha(\hat{d})$ .

*Testing the equality of the memory parameters in the case of independent samples.* We wish to test the null hypothesis  $d_1 = d_2$  against the alternative  $d_1 \neq d_2$  under the assumption that  $X_1$  and  $X_2$  are independent. The decision rule at  $\alpha$ -level of significance is the following: we reject the null hypothesis when

$$T_n > t_\alpha(\hat{d}). \quad (2.25)$$

The consistency of this test is ensured by Proposition 2.6(ii).

*Testing the equality of the memory parameters in the case of possibly dependent samples.* We wish to test the null hypothesis  $d_1 = d_2$  against the alternative  $d_1 > d_2$  in the general case when  $X_1$  and  $X_2$  are possibly dependent. The decision rule at  $\alpha$ -level of significance is the following: we reject the null hypothesis when

$$\tilde{T}_n > t_\alpha(\hat{d}). \quad (2.26)$$

The consistency of this test is ensured by Proposition 2.7(ii).

**Remark 2.9.** For testing  $d_1 = d_2$  against  $d_1 < d_2$ , the samples  $X_1$  and  $X_2$  should be exchanged in the statistic (2.18).

**Remark 2.10.** As noted by the referee, an undesirable feature of the testing procedure in (2.26) is that a rejection might occur not only when  $d_1 > d_2$  but also when  $d_1 < d_2$  (see Proposition 2.7(iii)). To alleviate this feature, in (2.26) one can use the statistic

$$\tilde{T}_n^+ = \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}}$$

instead of  $\tilde{T}_n$ . Note  $\tilde{T}_n = \tilde{T}_n^+ + (\tilde{T}_n^+)^{-1}$ . Under the assumptions of Proposition 2.7, the limit distribution of  $\tilde{T}_n^+$  can easily be obtained from the proof of this proposition. In particular, if  $d_1 = d_2 = d$ , then

$$\tilde{T}_n^+ \rightarrow_{\text{law}} \tilde{T}^+ = \frac{\hat{U}_1}{U_2},$$

where  $\hat{U}_1, U_2$  are independent and have the same distribution in (2.14). From the proof of Proposition 2.7, it also follows easily that  $\tilde{T}_n^+ \rightarrow_p \infty (d_1 > d_2)$  and  $\tilde{T}_n^+ \rightarrow_p \rho^2/(1 - \rho^2)(d_1 < d_2)$ , i.e.,  $\tilde{T}_n^+$  does not explode to infinity for  $d_1 < d_2$  and  $|\rho|$  not very close to 1.

### 2.3. Asymptotic behavior of the power function

We discuss in this section the asymptotic power of the testing procedures in (2.25) and (2.26).

For testing  $d_1 = d_2$  against  $d_1 > d_2$ , we have proved that the rejection probability of the null hypothesis tends to 1, i.e., that

$$P(\tilde{T}_n > t_\alpha(\hat{d})) \rightarrow 1, \quad (2.27)$$

for any  $\alpha \in (0, 1)$ .

Section 4 provides finite-sample rejection frequencies of the null hypothesis for some choices of parameters  $d_1, d_2$  and some bivariate FARIMA models. A natural question in this context is to estimate the convergence rate in (2.27), or the decay rate of the probability  $P(\tilde{T}_n \leq a)$ , as a function of  $a, n, d_1, d_2, \rho$  and (possibly) other quantities of the model assumptions.

From the proof of Proposition 2.7(ii), see (A.14) below, we have that for  $d_1 > d_2$ , the normalized statistic  $\tilde{T}_n$  has a limit distribution,

$$(q/n)^{2(d_1-d_2)}\tilde{T}_n = (q/n)^{2(d_1-d_2)}\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} + o_p(1) \rightarrow_{\text{law}} \frac{U_1}{(1 - \rho^2)U_2} = \Lambda, \quad (2.28)$$

say, where  $U_i, i = 1, 2$  are defined in (2.14). Therefore, we can expect that the probability  $P(\tilde{T}_n \leq a)$  decays as the probability  $P(\Lambda \leq a(q/n)^{2(d_1-d_2)})$ , when  $n \rightarrow \infty$ . The decay rate of  $P(\Lambda \leq x), x \rightarrow 0$  is unknown, even for independent  $U_1, U_2$ , but in principle can be estimated from Monte-Carlo experiments. It is also plausible that r.v.  $\Lambda$  has a bounded probability density near  $x = 0$  and so the above discussion suggests a decay rate  $P(\tilde{T}_n \leq a) = O((q/n)^{2(d_1-d_2)})$ .

However, the above argument is heuristic; in particular, the replacement of  $\tilde{T}_n$  by  $(q/n)^{2(d_1-d_2)}\Lambda$  is not rigorously justified. It is clear that in order to correctly assess the probability  $P(\tilde{T}_n \leq a)$ , it is necessary to control from above the probabilities of high values of  $\tilde{V}_1$  and  $S_{22,q}$  in the denominator of the ratio  $\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} = \frac{\tilde{V}_1 S_{22,q}}{V_2 \tilde{S}_{11,q}}$  and the probabilities of small values of  $V_2, \tilde{S}_{11,q}$  in the numerator of the last ratio. While the former probabilities can be controlled by the Markov inequality, direct estimation of the latter probabilities is difficult and is replaced by an assumption on the distribution functions of these r.v.'s, see (2.29) below.

**Proposition 2.11.** Let Assumption A( $d_1, d_2$ ) be satisfied with  $0 \leq d_2 < d_1 < 1/2$ . Moreover, assume that the distribution functions of the normalized r.v.'s  $n^{-2d_1}V_1$  and  $q^{-2d_2}S_{22,q}$  satisfy the following condition: for any  $M > 0, a_0 > 0$  there exists a constant  $K$  such that the inequalities

$$P(n^{-2d_1}V_1 \leq a) \leq Ka, \quad P(q^{-2d_2}S_{22,q} \leq a) \leq Ka \quad (2.29)$$

are satisfied for any  $n > M$ ,  $q > M$ ,  $n/q > M$  and any  $0 < a \leq a_0$ . Then there exists a constant  $K_1$ , independent of  $n$ ,  $q$ ,  $a$ , and such that

$$P(T_n \leq a) \leq K_1 a^{1/4} \left(\frac{q}{n}\right)^{(d_1-d_2)/2}, \quad P(\tilde{T}_n \leq a) \leq K_1 a^{1/4} \left(\frac{q}{n}\right)^{(d_1-d_2)/2} \quad (2.30)$$

hold for all  $n$ ,  $q$ ,  $n/q$  sufficiently large and any  $a \geq 0$  from a compact set.

The proof of the above proposition is given in [Appendix A.2](#). Note that condition (2.29) is implied by the existence of bounded probability densities of  $n^{-2d_1}V_1$  and  $q^{-2d_2}S_{22,q}$ . Also note that the assumptions of [Proposition 2.11](#) refer to *marginal* distributions of  $V_1$  and  $S_{22,q}$  only, and do not impose a restriction on the joint distribution of the four statistics in the definition of  $T_n$  and  $\tilde{T}_n$ .

### 3. Application to bivariate linear processes with long memory

In this section we specify [Assumptions A\( \$d\_1, d\_2\$ \)](#) and [B\( \$d\_1, d\_2\$ \)](#) to a class of bivariate linear models  $(X_1(t), X_2(t))$ ,  $t \in \mathbb{Z}$  as given by

$$X_i(t) = \sum_{k=0}^{\infty} \psi_{i1}(k) \xi_1(t-k) + \sum_{k=0}^{\infty} \psi_{i2}(k) \xi_2(t-k), \quad i = 1, 2, \quad (3.1)$$

where  $\psi_{ij}(k)$  are real coefficients with  $\sum_{k=0}^{\infty} \psi_{ij}^2(k) < \infty$  and  $(\xi_1(t), \xi_2(t))$ ,  $t \in \mathbb{Z}$  is a bivariate (weak) white noise with nondegenerate covariance matrix  $(\rho_{\xi,ij})_{i,j=1,2}$ . In other words,  $(\xi_1(t), \xi_2(t))$ ,  $t \in \mathbb{Z}$  is a sequence of random vectors with zero mean  $E\xi_1(t) = E\xi_2(t) = 0$  and covariances

$$E\xi_i(t)\xi_j(s) = \begin{cases} \rho_{\xi,ij}, & t = s, \\ 0, & t \neq s. \end{cases} \quad (3.2)$$

Without loss of generality, below we shall assume  $\rho_{\xi,11} = \rho_{\xi,22} = 1$ ,  $\rho_{\xi,12} = \rho_{\xi,21} = \rho_{\xi} \in (-1, 1)$ .

**Assumption  $\tilde{A}(d_{ij})$ .**  $(X_1(t), X_2(t))$  is a bivariate linear covariance stationary process as in (3.1) with coefficients  $\psi_{ij}(k)$  satisfying the following conditions:

- If  $d_{ij} \in (0, 1/2)$

$$\psi_{ij}(k) = (\alpha_{ij} + o(1)) |k|^{d_{ij}-1} \quad (k \rightarrow \infty)$$

where  $\alpha_{ij} \neq 0$  are some numbers,  $i, j = 1, 2$ .

- If  $d_{ij} = 0$

$$\sum_{k=0}^{\infty} |\psi_{ij}(k)| < \infty, \quad \alpha_{ij} = \sum_{k=0}^{\infty} \psi_{ij}(k).$$

**Assumption  $\tilde{B}(d_{ij})$ .** [Assumption  \$\tilde{A}\(d\_{ij}\)\$](#)  is satisfied and, moreover,  $(\xi_1(t), \xi_2(t))$ ,  $t \in \mathbb{Z}$  is a sequence of i.i.d. random vectors.

**Proposition 3.1.** (i) Let  $(X_1(t), X_2(t))$  satisfy [Assumption  \$\tilde{A}\(d\_{ij}\)\$](#) . Then the limits  $c_{ij}$  in [Assumption A\( \$d\_1, d\_2\$ \)](#), (2.5) exist, with

$$d_i = \max\{d_{i1}, d_{i2}\} \in [0, 1/2) \quad (i = 1, 2). \quad (3.3)$$

(ii) Let  $(X_1(t), X_2(t))$  satisfy [Assumption  \$\tilde{B}\(d\_{ij}\)\$](#)  and there exists  $\delta > 0$  such that  $E|\xi_i(t)|^{2+\delta} < \infty$  ( $i = 1, 2$ ). Then  $(X_1(t), X_2(t))$  satisfies [Assumptions A\( \$d\_1, d\_2\$ \)](#) and [B\( \$d\_1, d\_2\$ \)](#), with  $d_i$  as defined in (3.3). Moreover, the finite-dimensional convergence in [Assumption B\( \$d\_1, d\_2\$ \)](#), (2.11) extends to the functional convergence in the Skorohod space  $D[0, 1]$ .

**Remark 3.2.** [Proposition 3.1\(ii\)](#) complements Chung [6, Th. 1], who discussed convergence of partial sums of  $K$ -variate linear processes to  $K$ -variate fractional Brownian motion under slightly different assumptions. [Propositions 3.1\(i\)](#) and [2.2 \(2.7\)](#) also complement the result in [1] about consistency of the HAC estimator for linear processes, by relaxing the 4th moment condition on the noise in the case  $d_i \in [1/4, 1/2)$ .

Let us consider some parametric examples of bivariate linear processes. Hereafter, we denote by  $L$  the backward shift i.e.  $LX(t) = X(t-1)$ .



**Example 3.3.** Let  $a_{ij} \in \mathbb{R}$  ( $i = 1, 2$ ) be some constants, and let

$$X_i(t) = (1 - L)^{-d_i}(a_{i1}\xi_1(t) + a_{i2}\xi_2(t)) \quad (i = 1, 2) \quad (3.4)$$

be FARIMA(0,  $d_i$ , 0) processes, with possibly different parameters  $d_i \in (0, 1/2)$ . This process satisfies [Assumption  \$\tilde{A}\(d\_{ij}\)\$](#)  with  $d_{ij} = d_i$  and  $\alpha_{ij} = \Gamma(d_i)^{-1}a_{ij}$  ( $i, j = 1, 2$ ).

If  $(\xi_1(t), \xi_2(t))$  form a sequence of i.i.d. vectors as in [Assumption  \$\tilde{B}\(d\_{ij}\)\$](#) , partial sums of  $(X_1(t), X_2(t))$  converge to a bivariate fBm  $(B_{1,d_1}, B_{2,d_2})$ .

The limiting fBm has independent components if and only if the noise has uncorrelated components, i.e., if  $E(a_{11}\xi_1(t) + a_{12}\xi_2(t))(a_{21}\xi_1(t) + a_{22}\xi_2(t)) = 0$ . For  $d_1 = d_2$ , the last condition is equivalent to the uncorrelatedness of the components of the process:  $EX_1(t_1)X_2(t_2) = 0$  ( $t_1, t_2 \in \mathbb{Z}$ ).

**Example 3.4.** Consider the following system of linear equations:

$$(1 - L)^{d'_{11}}X_1(t) + \beta(1 - L)^{d'_{12}}X_2(t) = \xi_1(t),$$

$$(1 - L)^{d'_{22}}X_2(t) = \xi_2(t),$$

where  $d'_{ij} \in [0, 1/2)$ ,  $\beta \in \mathbb{R}$  are parameters,  $d'_{22} + d'_{11} - d'_{12} < 1/2$  and where  $(\xi_1(t), \xi_2(t))$ ,  $t \in \mathbb{Z}$  are as in (3.1). A covariance stationary solution of the above equation is given by

$$X_2(t) = (1 - L)^{-d'_{22}}\xi_2(t),$$

$$X_1(t) = (1 - L)^{-d'_{11}}\xi_1(t) - \beta(1 - L)^{d'_{12}-d'_{11}-d'_{22}}\xi_2(t).$$

Then  $(X_1(t), X_2(t))$  satisfies [Assumption  \$\tilde{A}\(d\_{ij}\)\$](#)  with  $d_{11} = d'_{11}$ ,  $d_{12} = d'_{22} + d'_{11} - d'_{12}$ ,  $d_{21} = 0$  and  $d_{22} = d'_{22}$ .

Assume  $(\xi_1(t), \xi_2(t))$  is a sequence of i.i.d. vectors satisfying the conditions in [Proposition 3.1\(ii\)](#) and let  $\beta \neq 0$ . There are three cases  $d'_{22} > d'_{12}$ ,  $d'_{22} < d'_{12}$  and  $d'_{22} = d'_{12}$  leading to  $d_{11} < d_{12}$ ,  $d_{11} > d_{12}$  and  $d_{11} = d_{12}$ , respectively. In each of these cases we can determine the memory parameter  $d_i$  of  $(X_i(t))$ ,  $i = 1, 2$  and the limiting bi-fBm in [Assumption B\( \$d\_1, d\_2\$ \)](#), together with the correlation coefficient  $\rho$ .

#### 4. A simulation study

In this section we assess the finite-sample performance of our procedures to test  $d_1 = d_2$  versus  $d_1 > d_2$  and provide a practical recommendation for the choice of the bandwidth parameter  $q$ .

The memory parameters  $d_1$  and  $d_2$  are estimated with the help of the adaptive version of the FEXP estimator (see [14]), which in practice turns out to be less sensitive to the short memory part of the long memory process as compared to other estimators. The bandwidth parameter is chosen according to the adaptive formula (A.32) derived in [Appendix](#) (see [Appendix A.3](#)). The optimisation of the bandwidth is realised under the null hypothesis in order to ensure a good size to the test procedure. The choice of  $q$  depends on  $\hat{d} = \frac{1}{2}(\hat{d}_1 + \hat{d}_2)$  as in the expansion obtained by Abadir et al. [1, (2.14)], but also takes into account the short memory spectrum, in a form of certain coefficient.

The simulated samples are independent or dependent Gaussian FARIMA processes with different fractional and autoregressive/moving average parameters. The 5% quantile function in (2.22) was approximated from extensive Monte-Carlo experiments by

$$t_{5\%}(d) \approx 3.7d^2 + 8.6d + 5.2. \quad (4.1)$$

**Independent samples.** [Tables 1–3](#) concerns the case of independent samples, the test procedure is based on  $T_n$ . [Tables 1 and 2](#) report the percentages of rejection of the null hypothesis  $d_1 = d_2$  of the test  $T_n > t_{5\%}(\hat{d})$  from 1000 replications of independent FARIMA(1,  $d$ , 0) samples of size  $n \in \{1024, 4096\}$ , for five values of  $d_i \in \{0, 0.1, 0.2, 0.3, 0.4\}$  and three values  $a_i \in \{0, 0.4, 0.8\}$  of the autoregressive parameter. Recall from [Proposition 2.6](#) that for independent samples  $T_n$  and  $\tilde{T}_n$  have the same limiting distribution. We can see from both tables that the  $T_n$  test has fairly good size for most values of  $d_i$  and  $a_i$ . [Table 3](#) provides the mean values of  $\hat{q}$ . Since these values are rather scattered across the table and the rejection frequency is very sensitive to the choice of  $q$ , the general impression from [Tables 1–3](#) is that the adaptive choice of  $q$  in (A.32) was necessary. We also note that the power of the test decreases with increase of  $|a_1 - a_2|$ . The last fact can possibly be explained by the bias induced by the AR part in the FEXP estimator of  $d_i$ .

**Dependent samples.** [Table 4](#) reports the performance of the test  $\tilde{T}_n > t_{5\%}(\hat{d})$  on dependent samples as in [Example 3.3](#), with  $a_{11} = a_{22} = 1 - p$ ,  $a_{12} = a_{21} = p$ , where  $p \in [0, 1/2)$  is a parameter. In other words,  $X_i$  are FARIMA(0,  $d_i$ , 0) processes with mutually correlated innovations. The asymptotic correlation coefficient  $\rho$  between normalized partial sums of  $X_1$  and  $X_2$  is proportional to  $2p(1 - p)/(p^2 + (1 - p)^2)$  and so  $\rho$  increases monotonically from 0 to 1 as  $p$  increases from 0 to 1/2. Since  $p = 0$  corresponds to independent samples, the results in [Table 4](#) can be compared with those for  $a_1 = a_2 = 0$  in [Tables 1 and 2](#). It appears that both tests  $T_n$  and  $\tilde{T}_n$  perform similarly and that the long-run parameter  $\rho$  is well estimated to be zero.



**Table 1**

Frequency of rejection (in percentages) of the null hypothesis of the test  $T_n > c_{5\%}(\hat{d})$ . The samples are simulated following FAR(1,  $d_i$ ) models. For fixed  $a_1, a_2$ , each cell contains a triangular array of dimension  $5 \times 5$  corresponding to the different parameters ( $d_1, d_2$ ) with  $d_i \in \{0, 0.1, 0.2, 0.3, 0.4\}$  ( $i = 1, 2$ ) and  $d_1 \geq d_2$ . The sample size is  $n = 1024$ . The estimation is based on 1000 replications.

$n = 1024$		$a_2 = 0$					$a_2 = 0.4$					$a_2 = 0.8$				
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4										
$a_1 = 0$	0	4.1														
	0.1	12	5.0													
	0.2	36	13	3.8												
	0.3	64	32	9.3	3.2											
	0.4	84	55	26	7.4	3.6										
$a_1 = 0.4$	0	5.8					4.5									
	0.1	9.1	4.5				13	3.7								
	0.2	21	7.4	4.0			35	9.6	4.9							
	0.3	45	20	6.6	3.7		59	33	12	3.7						
	0.4	66	35	17	4.2	3.4	82	58	26	9.1	2.5					
$a_1 = 0.8$	0	3.8					4.9					2.2				
	0.1	3.4	3.6				4.0	4.4				9.1	4.5			
	0.2	13	3.7	3.9			13	4.4	4.8			28	9.9	3.8		
	0.3	26	10	3.6	5.2		32	14	4.0	4.4		56	31	11	3.2	
	0.4	45	25	9.6	4.5	5.9	55	32	13	4.1	5.8	80	55	31	9.5	3.5

**Table 2**

The same results as Table 1 for sample size  $n = 4096$ .

$n = 4096$		$a_2 = 0$					$a_2 = 0.4$					$a_2 = 0.8$				
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4										
$a_1 = 0$	0	5.1														
	0.1	23	4.1													
	0.2	59	20	4.8												
	0.3	86	49	14	4.3											
	0.4	96	77	42	11	2.9										
$a_1 = 0.4$	0	5.2					3.5									
	0.1	11	4.2				20	5.7								
	0.2	35	11	4.8			57	19	4.2							
	0.3	70	34	11	5.2		84	54	16	4.0						
	0.4	88	64	30	10	3.3	95	78	48	14	3.0					
$a_1 = 0.8$	0	5.4					5.2					4.9				
	0.1	7.4	3.8				9.0	4.3				20	5.5			
	0.2	25	9.0	4.3			30	10	4.8			42	15	3.8		
	0.3	53	24	7.5	4.5		59	26	10	3.8		84	47	16	4.5	
	0.4	75	44	21	6.2	4.2	82	50	25	8.2	5.3	95	79	50	16	4.2

The purpose of Tables 5 and 6 is twofold. Firstly, we want to evaluate the performance of  $\tilde{T}_n$  on independent samples. Secondly, we want to assess the robustness of the adaptive formula (A.32) for bandwidth based on AR approximation of the short memory part with respect to other short memory specifications. To this end, we generate a FARIMA(3,  $d$ , 0) process with polynomial AR function  $1 + 0.7x^3$  (Table 5) and a FARIMA(0,  $d$ , 2) process with polynomial MA function  $1 - (1/6)x + (1/6)x^2$  (Table 6). Together with the zero hypothesis rejection frequencies, Tables 5 and 6 also provide the (averaged) values of the adaptive estimator of the bandwidth  $q$ . One can immediately recognize that the estimated values of  $q$  in Table 5 are much greater than the corresponding values on Table 6; nevertheless the size of the  $\tilde{T}_n$  test is respected in both tables. One can conclude from Table 6 that the adaptive formula for  $q$  works rather well even if the FAR model (on which this formula is based) is misspecified.

## 5. Concluding remarks

The paper constructs a two-sample test for comparison of long memory parameters  $d_i \in [0, 1/2)$ ,  $i = 1, 2$  of covariance-stationary time series  $X_i$ ,  $i = 1, 2$  with discrete time. The test statistic,  $T_n$ , is defined as the sum of the ratio and the reciprocal ratio of the rescaled variance ( $V/S$ ) statistics, computed for each sample, whose asymptotic and finite-sample behavior was studied in [11,12]. Under some assumptions which involve the existence of long-run covariances and the joint convergence

**Table 3**The mean values on 1000 replications of  $\hat{q}$  according to (A.32) for the simulations of Table 2.

$n = 4096$		$a_2 = 0$					$a_2 = 0.4$					$a_2 = 0.8$				
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4										
$a_1 = 0$	0	3.2														
	0.1	2.7	2.1													
	0.2	2.3	2.0	1.7												
	0.3	1.9	1.8	1.4	1.0											
	0.4	1.8	1.5	1.0	0.5	0.3										
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4					
$a_1 = 0.4$	0	11.2					5.4									
	0.1	9.0	7.5				4.4	3.7								
	0.2	7.5	6.3	5.3			3.6	3.0	2.7							
	0.3	6.2	5.3	4.3	2.9		3.1	2.6	2.0	1.4						
	0.4	5.3	4.4	2.9	1.8	1.0	2.6	2.0	1.4	0.8	0.4					
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
$a_1 = 0.8$	0	24.8					22.0					10.2				
	0.1	19.9	16.2				17.5	14.1				8.2	6.7			
	0.2	16.2	13.4	11.3			14.2	11.6	9.6			6.7	5.7	4.7		
	0.3	13.5	11.3	8.8	5.9		11.6	9.7	7.6	4.9		5.6	4.6	3.4	2.3	
	0.4	11.3	8.8	5.8	3.6	2.1	9.7	7.5	5.0	3.0	1.6	4.5	3.4	2.2	1.3	0.6

**Table 4**Frequency of rejection (in percentages) of the null hypothesis of the test  $\tilde{T}_n > c_{5\%}(\hat{d})$ . The samples are simulated following the model in (3.4). For fixed  $p$ , each cell contains a triangular array of dimension  $5 \times 5$  corresponding to the different parameters  $(d_1, d_2)$  with  $d_i \in \{0, 0.1, 0.2, 0.3, 0.4\}$  ( $i = 1, 2$ ) and  $d_2 \leq d_1$ .

		$n = 1028$					$n = 4096$				
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
$p = 0$	0	4.3					5.6				
	0.1	14	5.4				22	6.4			
	0.2	38	13	3.7			62	17	5.7		
	0.3	66	33	8.5	3.9		87	55	17	5.9	
	0.4	83	57	27	7.5	3.4	97	83	45	13	3.8
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
$p = 0.15$	0	5.8					5.8				
	0.1	13	4.9				22	5.9			
	0.2	40	10	6.2			64	21	6.3		
	0.3	69	35	9.6	3.7		90	58	17	5.9	
	0.4	84	61	26	6.0	3.0	98	83	50	14	4.1
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
$p = 0.35$	0	5.4					4.9				
	0.1	17	4.5				30	5.3			
	0.2	54	14	4.7			84	26	5.3		
	0.3	81	43	9.0	3.5		98	76	20	4.2	
	0.4	95	74	30	7.2	2.9	100	95	60	13	3.1
	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
$p = 0.45$	0	6.2					5.5				
	0.1	43	3.0				84	4.6			
	0.2	84	28	3.6			98	63	5.9		
	0.3	95	74	13	3.9		100	96	33	3.1	
	0.4	97	90	46	6.8	3.2	100	99	85	14	4.4

**Table 5**[Left] Frequency of rejection (in percentages) of the null hypothesis of the test  $\tilde{T}_n > c_{5\%}(\hat{d})$ . The two samples  $X_1$  and  $X_2$  are independent.  $X_1$  is simulated from FARIMA(3,  $d_1$ , 0) model with polynomial AR function  $1 + 0.7x^3$  and  $X_2$  from FARIMA(0,  $d_2$ , 0) model. [Right] Adaptive estimation of the bandwidth parameter  $q$ . The sample size is 4096 and the statistics are evaluated from 1000 independent replications.

$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4
0	6.1					0	34.7				
0.1	7.4	5.6				0.1	28.4	24.0			
0.2	21	8.2	4.0			0.2	23.9	20.8	18.2		
0.3	46	20	7.5	6.0		0.3	20.8	18.3	16.0	13.0	
0.4	64	39	18	6.0	4.5	0.4	18.2	15.9	13.0	9.0	5.8

**Table 6**

[Left] Frequency of rejection (in percentages) of the null hypothesis of the test  $\tilde{T}_n > c_{5\%}(\hat{d})$ . The two samples  $X_1$  and  $X_2$  are independent.  $X_1$  is simulated from FARIMA(0,  $d_1$ , 2) model with polynomial MA function  $1 - (1/6)x + (1/6)x^2$  and  $X_2$  from FARIMA(0,  $d_2$ , 0) model. [Right] Adaptive estimation of the bandwidth parameter  $q$ . The sample size is 4096 and the statistics are evaluated from 1000 independent replications.

$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4	$d_1 \setminus d_2$	0	0.1	0.2	0.3	0.4
0	4.9					0	7.0				
0.1	17	6.7				0.1	6.1	5.2			
0.2	48	16	4.9			0.2	5.3	4.6	4.3		
0.3	72	44	15	4.3		0.3	4.8	4.2	3.8	2.8	
0.4	88	71	35	11	4.3	0.4	4.3	3.5	2.7	1.8	1.1

of partial sums of  $X_1$  and  $X_2$  to a bivariate fractional Brownian motion, we derive the asymptotic null distribution of  $T_n$ . A modification  $\tilde{T}_n$  of the test statistic  $T_n$  is introduced and shown to be asymptotically free of the long-run correlation coefficient between the two samples. The case when  $(X_1, X_2)$  form a bivariate linear process is discussed in detail. Simulation results using FARIMA samples with various fractional and autoregressive parameters show that the proposed tests have a good size for most values of fractional and autoregressive/moving average parameters. The robustness property of the test is largely due to our choice of bandwidth according to the adaptive formula in (A.32) which takes into account the estimated difference of short memory spectrum of the sampled processes. The derivation of the last formula uses the asymptotic expansion of the HAC estimator in [1].

### Acknowledgments

The authors are grateful to the referee and associated editor for comments and suggestions which helped to improve the original version of the paper. The joint work was partially supported by the research project MATPYL of the Fédération de Mathématiques des Pays de Loire. The research of the third author (D.S.) was partially supported by the Lithuanian State Science and Studies Foundation grant no. T-70/09.

### Appendix. Proofs and auxiliary results

Appendix A.1 provides alternative definition of bi-fBm by explicit cross-covariance function. Appendix A.2 contains proofs of Propositions 2.2, 2.6, 2.7, 2.11 and 3.1. Appendix A.3 is given to the derivation of the adaptive bandwidth formula in (A.32).

#### A.1. Covariance function of bivariate fractional Brownian motion

From (2.9) and [20] we have for any  $s, t \in \mathbb{R}$

$$EB_i(s)B_i(t) = \frac{1}{2} \left\{ |s|^{2d_i+1} + |t|^{2d_i+1} - |t-s|^{2d_i+1} \right\}, \quad i = 1, 2.$$

The analytic expression of cross-covariance  $EB_1(s)B_2(t)$  is derived in [16]. It takes a different form in the cases  $d_1 + d_2 \neq 0$  and  $d_1 + d_2 = 0$ . Let

$$\psi(d_1, d_2) = \frac{B(d_1 + 1, d_2 + 1) \sqrt{\cos(d_1 \pi) \cos(d_2 \pi)}}{\sqrt{B(d_1 + 1, d_1 + 1) B(d_2 + 1, d_2 + 1)}}.$$

Let  $d_1 + d_2 \neq 0$ ,  $d_1, d_2 \in (-1/2, 1/2)$ . Then

$$EB_1(s)B_2(t) = \frac{\rho_W}{2} \left\{ g_{12}(s)|s|^{d_1+d_2+1} + g_{21}(t)|t|^{d_1+d_2+1} - g_{21}(t-s)|t-s|^{d_1+d_2+1} \right\}, \quad (\text{A.1})$$

where

$$g_{ij}(t) = \begin{cases} g_{ij}, & t > 0, \\ g_{ji}, & t < 0 \end{cases}$$

and where

$$g_{12} = \psi(d_1, d_2) \sin(d_1 \pi) / \sin((d_1 + d_2) \pi), \quad g_{21} = \psi(d_1, d_2) \sin(d_2 \pi) / \sin((d_1 + d_2) \pi). \quad (\text{A.2})$$

In the case  $d_1 + d_2 = 0$ ,

$$EB_1(s)B_2(t) = \frac{\rho_W}{2} \left\{ g_1(|s| + |t| - |t-s|) + g_2(t \log |t| + s \log |s| - (t-s) \log |t-s|) \right\}, \quad (\text{A.3})$$

where

$$g_1 = (1/2) \psi(d_1, d_2) (\cos(\pi d_1) + \cos(\pi d_2)), \quad g_2 = \psi(d_1, d_2) (d_2 - d_1). \quad (\text{A.4})$$

## A.2. Proofs of the propositions

**Proof of Proposition 2.2.** The first relation in (2.7) is immediate from (2.5) and (2.6); see also [12, (3.4)]. Then, the second relation in (2.7) follows from (2.8), which is proved below.

Assume without loss of generality that  $\mu_i = \mu_j = 0$ . By Assumption A( $d_1, d_2$ ), there exists a constant  $C$  such that for any  $n, h \geq 1$ ,

$$\mathbb{E} \left( \sum_{t=n-h+1}^n X_i(t) \right)^2 \leq Ch^{1+2d_i}, \quad i = 1, 2. \quad (\text{A.5})$$

In particular,  $\mathbb{E}\bar{X}_i^2 = O(n^{2d_i-1})$ . Let  $h \geq 1$ . Then

$$\hat{\gamma}_{ij}(h) - \hat{\gamma}_{ij}^\circ(h) = -\bar{X}_i\bar{X}_j + \bar{X}_i \frac{1}{n} \sum_{t=n-h+1}^n X_j(t) + \bar{X}_j \frac{1}{n} \sum_{t=n-h+1}^n X_i(t) - \frac{h}{n} \bar{X}_i\bar{X}_j.$$

Clearly, (2.8) follows from  $q/n \rightarrow 0$  and

$$\frac{1}{qn} |\bar{X}_i| \sum_{h=1}^q \left| \sum_{t=n-h+1}^n X_j(t) \right| = o_p(n^{d_i+d_j-3/2}). \quad (\text{A.6})$$

By (A.5) and Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E} |\bar{X}_i| \sum_{h=1}^q \left| \sum_{t=n-h+1}^n X_j(t) \right| &\leq \mathbb{E}^{1/2} \bar{X}_i^2 \mathbb{E}^{1/2} \sum_{h=1}^q \left| \sum_{t=n-h+1}^n X_j(t) \right|^2 \\ &\leq Cn^{d_i-1/2} \left( \sum_{h=1}^q h^{1+2d_j} \right)^{1/2} \\ &= qn o(n^{d_i+d_j-3/2}) \end{aligned}$$

since  $q = o(n)$ . This proves (A.6).  $\square$

**Proof of Proposition 2.6.** Both parts (i) and (ii) follow from the joint convergence

$$((q/n)^{2d_1} V_1 / S_{11,q}, (q/n)^{2d_2} V_2 / S_{22,q}) \rightarrow_{\text{law}} (U_1, U_2), \quad (\text{A.7})$$

with  $U_1, U_2$  as in (2.14). The last relation follows similarly as in [12, Lemmas 7.1 and 7.2].  $\square$

**Proof of Proposition 2.7.** (i) Recall from Remark 2.4 that  $\rho = \rho_W$ . From (2.7), (2.17) and Assumptions A( $d_1, d_2$ ) and B( $d_1, d_2$ ) (with  $d_1 = d_2 = d$ ) it follows that

$$\begin{aligned} q^{-2d} \tilde{S}_{11,q} &= q^{-2d} S_{11,q} - \frac{(q^{-2d} S_{12,q})^2}{q^{-2d} S_{22,q}} \\ &\rightarrow_p c_{11} - \frac{c_{12}^2}{c_{22}} = c_{11}(1 - \rho^2) \end{aligned}$$

and

$$\begin{aligned} n^{-d-(1/2)} \left( \sum_{t=1}^{\lfloor n\tau \rfloor} (\tilde{X}_1(t) - \tilde{\bar{X}}_1), \sum_{t=1}^{\lfloor n\tau \rfloor} (X_2(t) - \bar{X}_2) \right) &\rightarrow_{\text{fdd}} (\sqrt{c_{11}}(B_1(\tau) - \rho B_2(\tau)), \sqrt{c_{22}}B_2(\tau)) \\ &=_{\text{fdd}} (\sqrt{c_{11}(1 - \rho^2)}\hat{B}_1(\tau), \sqrt{c_{22}}B_2(\tau)), \end{aligned}$$

where  $(\hat{B}_1(\tau) = (B_1(\tau) - \rho B_2(\tau))/\sqrt{1 - \rho^2})$  is a fBm independent of  $(B_2(\tau))$ ; see Remark 2.4. Let  $(\hat{B}_1^0(\tau) = \hat{B}_1(\tau) - \tau \hat{B}_1(1), \tau \in [0, 1])$ ,  $(B_2^0(\tau) = B_2(\tau) - \tau B_2(1), \tau \in [0, 1])$  be respective fractional Brownian bridges.

These relations together with (2.7) imply similarly as in [12] that

$$n^{-2d} \tilde{V}_1 \rightarrow_{\text{law}} c_{11}(1 - \rho^2) \left( \int_0^1 (\hat{B}_1^0(\tau))^2 d\tau - \left( \int_0^1 \hat{B}_1^0(\tau) d\tau \right)^2 \right), \quad (\text{A.8})$$

$$n^{-2d} V_2 \rightarrow_{\text{law}} c_{22} \left( \int_0^1 (B_2^0(\tau))^2 d\tau - \left( \int_0^1 B_2^0(\tau) d\tau \right)^2 \right), \quad (\text{A.9})$$

$$q^{-2d}\tilde{S}_{11,q} \rightarrow_{\text{law}} c_{11}(1 - \rho^2), \quad (\text{A.10})$$

$$q^{-2d}S_{22,q} \rightarrow_{\text{law}} c_{22} \quad (\text{A.11})$$

as  $n, q, n/q \rightarrow \infty$ , as well as the *joint* convergence of the four quantities in (A.8)–(A.11). Since the limits in (A.8)–(A.11) are a.s. strictly positive and  $(\hat{B}_1(\tau))$  is independent of  $(B_2(\tau))$ , this proves (2.19) and part (i).

(ii) From (2.7), (2.17) we have

$$\begin{aligned} q^{-2d_1}\tilde{S}_{11,q} &= q^{-2d_1}S_{11,q} - \frac{(q^{-d_1-d_2}S_{12,q})^2}{q^{-2d_2}S_{22,q}} \\ &\rightarrow_p c_{11} - \frac{c_{12}^2}{c_{22}} = c_{11}(1 - \rho^2). \end{aligned} \quad (\text{A.12})$$

From (2.7) and Assumption B( $d_1, d_2$ ) we obtain that

$$\begin{aligned} \frac{1}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (\tilde{X}_1(t) - \bar{X}_1) &= \frac{1}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (X_1(t) - \bar{X}_1) - \frac{q^{-d_1-d_2}S_{12,q}}{q^{-2d_2}S_{22,q}} \left(\frac{q}{n}\right)^{d_1-d_2} \frac{1}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (X_2(t) - \bar{X}_2) \\ &\rightarrow_{\text{fdd}} \sqrt{c_{11}}B_1(\tau). \end{aligned} \quad (\text{A.13})$$

Using similar arguments as in part (i), from (A.12) and (A.13) we get

$$\left( (q/n)^{2(d_1-d_2)} \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}}, (q/n)^{2(d_2-d_1)} \frac{V_2/S_{22,q}}{\tilde{V}_1/\tilde{S}_{11,q}} \right) \rightarrow_{\text{law}} \left( \frac{U_1}{(1-\rho^2)U_2}, \frac{(1-\rho^2)U_2}{U_1} \right), \quad (\text{A.14})$$

where  $U_i, i = 1, 2$  are defined in (2.14). Clearly, (A.14) implies (2.20) and part (ii).

(iii) In this case, (A.12) is again valid but (A.13) must be changed to

$$\begin{aligned} \frac{q^{d_2-d_1}}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (\tilde{X}_1(t) - \bar{X}_1) &= \frac{(q/n)^{d_2-d_1}}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (X_1(t) - \bar{X}_1) - \frac{q^{-d_1-d_2}S_{12,q}}{q^{-2d_2}S_{22,q}} \frac{1}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (X_2(t) - \bar{X}_2) \\ &\rightarrow_{\text{fdd}} -\frac{c_{12}}{\sqrt{c_{22}}}B_2(\tau). \end{aligned} \quad (\text{A.15})$$

From (A.15) we obtain

$$\frac{q^{2(d_2-d_1)}}{n^{2d_2}}\tilde{V}_1 \rightarrow_{\text{law}} c_{11}\rho^2 \left( \int_0^1 (B_2^0(\tau))^2 d\tau - \left( \int_0^1 B_2^0(\tau) d\tau \right)^2 \right).$$

Combining this result with (A.12) and the convergences in (A.9), (A.11), with  $d = d_2$ , one obtains

$$\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} \rightarrow_{\text{law}} \frac{\rho^2}{1 - \rho^2},$$

proving (2.21).  $\square$

**Proof of Proposition 2.11.** We shall prove the second inequality in (2.30) only since the first one can be proved analogously. We shall use the following elementary inequalities: for any r.v.  $\xi, \eta \geq 0$  and any  $x > 0$

$$P\left(\frac{\xi}{\eta} \leq x\right) \leq P(\xi \leq \sqrt{x}) + P\left(\eta > \frac{1}{\sqrt{x}}\right), \quad (\text{A.16})$$

$$P(\xi\eta \leq x) \leq P(\xi \leq \sqrt{x}) + P(\eta \leq \sqrt{x}), \quad P(\xi - \eta \leq x) \leq P(\xi \leq 2x) + P(\eta > x).$$

Denote

$$\begin{aligned} \tilde{\xi}_1 &= n^{-2d_1}\tilde{V}_1, & \xi_1 &= n^{-2d_1}V_1, & \xi_2 &= q^{-2d_2}S_{22,q}, & \tilde{\xi}_3 &= q^{-2d_1}\tilde{S}_{11,q}, \\ \xi_3 &= q^{-2d_1}S_{11,q}, & \xi_4 &= n^{-2d_2}V_2, & x &= a\left(\frac{q}{n}\right)^{2(d_1-d_2)}. \end{aligned}$$

Then

$$\begin{aligned} P(\tilde{T}_n \leq a) &\leq P\left(\frac{\tilde{\xi}_1 \xi_2}{\tilde{\xi}_3 \xi_4} \leq x\right) \\ &\leq P\left(\tilde{\xi}_1 \xi_2 \leq x^{1/2}\right) + P\left(\tilde{\xi}_3 \xi_4 > x^{-1/2}\right) \\ &\leq P\left(\tilde{\xi}_1 \leq x^{1/4}\right) + P\left(\xi_2 \leq x^{1/4}\right) + P\left(\tilde{\xi}_3 > x^{-1/4}\right) + P\left(\xi_4 > x^{-1/4}\right). \end{aligned} \quad (\text{A.17})$$

Next, using the inequality  $\tilde{V}_1 \geq (1/2)V_1 - \hat{\beta}^2 V_2 = (1/2)V_1 - \frac{\hat{\rho}^2 S_{11,q}}{S_{22,q}} V_2$  and the facts that  $|\hat{\rho}| \leq 1$ ,  $V_2 \geq 0$ , we get

$$\tilde{\xi}_1 \geq (1/2)\xi_1 - \eta_1, \quad (\text{A.18})$$

where

$$\eta_1 = \frac{S_{11,q}}{S_{22,q}} (n^{-2d_1} V_2) = x \frac{\xi_3 \xi_4}{\xi_2}.$$

Relations (A.18) and (A.16) yield

$$\begin{aligned} P\left(\tilde{\xi}_1 \leq x^{1/4}\right) &\leq P\left((1/2)\xi_1 - \eta_1 \leq x^{1/4}\right) \\ &\leq P(\xi_1 \leq 4x^{1/4}) + P(\eta_1 > x^{1/4}) \\ &= P(\xi_1 \leq 4x^{1/4}) + P\left(x \frac{\xi_3 \xi_4}{\xi_2} > x^{1/4}\right) \\ &\leq P(\xi_1 \leq 4x^{1/4}) + P(\xi_3 > x^{-1/4}) + P(\xi_4 > x^{-1/4}) + P(\xi_2 \leq x^{1/4}). \end{aligned} \quad (\text{A.19})$$

Combining (A.17) and (A.19) and using  $\tilde{S}_{11,q} \geq S_{11,q}$ , see (2.17), we obtain

$$P(\tilde{T}_n \leq a) \leq P(\xi_1 \leq 4x^{1/4}) + 2P(\xi_2 \leq x^{1/4}) + 2P(\xi_3 > x^{-1/4}) + 2P(\xi_4 > x^{-1/4}). \quad (\text{A.20})$$

From Assumption A( $d_1, d_2$ ) and (A.30) we obtain

$$\begin{aligned} ES_{11,q} &= \frac{1}{q+1} E\left(\sum_{t=0}^q X_1(t)\right)^2 + E \sum_{|h| \leq q} \left(1 - \frac{|h|}{q+1}\right) (\hat{\gamma}_{11}(h) - \hat{\gamma}_{11}^\circ(h)) \\ &\leq K_2 q^{2d_1}, \\ EV_2 &\leq K_3 n^{2d_2} \end{aligned} \quad (\text{A.21})$$

for some constants  $K_2, K_3$  independent of  $n, q$ , implying  $E\xi_3 \leq K_2$ ,  $E\xi_4 \leq K_3$ . From (A.20) and (A.21), the Markov inequality, and assumption (2.29), the statement of the proposition easily follows.  $\square$

**Proof of Proposition 3.1.** With exception of (2.6), all other facts in the statement of the proposition follow similarly or using the argument developed in [8,3,12] and other papers. In particular, the joint convergence of partial sums of  $(X_1, X_2)$  can be proved by using the scheme of discrete stochastic integrals in [22]. See also [6, Proof of Theorem 1].

Let us prove the convergence of empirical long-run covariances in (2.6) or, equivalently, in (2.7). Denote

$$X_{ij}(t) = \sum_{k=0}^{\infty} \psi_{ij}(k) \xi_j(t-k), \quad i, j = 1, 2. \quad (\text{A.22})$$

It suffices to show the convergence of the HAC estimates of long-run covariances  $c_{ij,i'j'}$  of components  $X_{ij}$  and  $X_{i'j'}$  in (A.22), for any pairs  $(i, j), (i', j') \in \{1, 2\} \times \{1, 2\}$ ; more precisely, to show that

$$q^{-d_{ij}-d_{i'j'}} S_{ij,i'j',q} \rightarrow_p c_{ij,i'j'} \quad (i, j, i', j' = 1, 2), \quad (\text{A.23})$$

where  $S_{ij,i'j',q}$  is defined as in (2.1) with  $\hat{\gamma}_{ij}(h)$  replaced by the empirical covariance  $\hat{\gamma}_{ij,i'j'}(h)$  between observations  $X_{ij}(t)$ ,  $t = 1, \dots, n$  and  $X_{i'j'}(t)$ ,  $t = 1, \dots, n$ .

Fix  $i, j, i', j' \in \{1, 2\}$  and denote  $X(t) = X_{ij}(t)$ ,  $X'(t) = X_{i'j'}(t)$ ,  $d = d_{ij}$ ,  $d' = d_{i'j'}$ ,

$$S_q = S_{ij,i'j',q}, \quad \gamma(h) = \gamma_{ij,i'j'}(h), \quad \hat{\gamma}(h) = \hat{\gamma}_{ij,i'j'}(h), \quad c = c_{ij,i'j'},$$

$\psi(k) = \psi_{ij}(k)$ ,  $\psi'(k) = \psi_{i'j'}(k)$ ,  $\xi(s) = \xi_j(s)$ ,  $\xi'(s) = \xi_{j'}(s)$ ,  $\tilde{\rho}_\xi = \rho_{\xi,jj'}$  for short. Write  $S_q = S'_q + S''_q$ , where

$$S'_q = \sum_{h=-q}^q \left(1 - \frac{|h|}{q+1}\right) \tilde{\gamma}(h), \quad S''_q = \sum_{h=-q}^q \left(1 - \frac{|h|}{q+1}\right) (\hat{\gamma}(h) - \tilde{\gamma}(h)),$$

where

$$\tilde{\gamma}(h) = n^{-1} \begin{cases} \sum_{t=1}^{n-h} X(t)X'(t+h), & h \geq 0, \\ \sum_{t=1-h}^n X(t)X'(t+h), & h \leq 0 \end{cases}$$

is the empirical covariance from noncentered observations; c.f. (2.2). Then (A.23) follows from

$$q^{-d-d'} S'_q \rightarrow_p c, \quad S''_q = o_p(q^{d+d'}). \quad (\text{A.24})$$

In the subsequent proof of the first relation of (A.24), we first assume  $d > 0, d' > 0$ . Split  $S'_q = \tilde{\gamma}(0) + \sum_{h=-q}^{-1} \left(1 - \frac{|h|}{q+1}\right) \tilde{\gamma}(h) + \sum_{h=1}^q \left(1 - \frac{|h|}{q+1}\right) \tilde{\gamma}(h)$ . Here, the last two sums can be treated similarly and  $\tilde{\gamma}(0) = O_p(1)$  is negligible. Consider

$$\sum_{h=1}^q \left(1 - \frac{h}{q+1}\right) \tilde{\gamma}(h) = \sum_{i=1}^3 U_i,$$

where

$$\begin{aligned} U_1 &= \tilde{\rho}_\xi \sum_s \sum_{h=1}^q \left(1 - \frac{h}{q+1}\right) \frac{1}{n} \sum_{t=1}^{n-h} \psi(t-s) \psi'(t+h-s), \\ U_2 &= \sum_s \eta_s \sum_{h=1}^q \left(1 - \frac{h}{q+1}\right) \frac{1}{n} \sum_{t=1}^{n-h} \psi(t-s) \psi'(t+h-s), \end{aligned} \quad (\text{A.25})$$

$$U_3 = \sum_{s \neq s'} \sum_{h=1}^q \left(1 - \frac{h}{q+1}\right) \frac{1}{n} \sum_{t=1}^{n-h} \psi(t-s) \psi'(t+h-s') \xi(s) \xi'(s'), \quad (\text{A.26})$$

where  $\eta_s = \xi(s) \xi'(s) - E\xi(s) \xi'(s) = \xi(s) \xi'(s) - \tilde{\rho}_\xi$ , the sums  $\sum_s$  and  $\sum_{s \neq s'}$  are taken over all  $s \in \mathbb{Z}$  and  $s, s' \in \mathbb{Z}, s \neq s'$ , respectively, and where we put  $\psi(t) = \psi'(t) = 0$  ( $t < 0$ ).

First, consider the (nonrandom) term  $U_1$ . Using the asymptotics of  $\psi$  and  $\psi'$  and the dominated convergence theorem, we easily obtain that, as  $q \rightarrow \infty, n \rightarrow \infty, q/n \rightarrow 0$ ,

$$\begin{aligned} U_1 &= \tilde{\rho}_\xi \sum_{h=1}^q \left(1 - \frac{h}{q+1}\right) \left( \frac{1}{n} \sum_{t=1}^{n-h} 1 \right) \sum_{k=0}^{\infty} \psi(k) \psi'(h+k) \\ &\sim \tilde{\rho}_\xi \alpha \alpha' \sum_{h=1}^q \int_0^{\infty} k^{d-1} (h+k)^{d'-1} dk \\ &\sim \tilde{\rho}_\xi \alpha \alpha' B(d, 1-d-d') \sum_{h=1}^q h^{d+d'-1} \\ &\sim \frac{1}{2} c q^{d+d'}, \end{aligned}$$

where  $c = 2\tilde{\rho}_\xi \alpha \alpha' B(d, 1-d-d')/(d+d')$ . Then, the first relation in (A.24) follows from

$$q^{-d-d'} U_i = o_p(1) \quad (i = 2, 3). \quad (\text{A.27})$$

To estimate  $U_2$ , we use the fact that  $(\eta(s), s \in \mathbb{Z})$  are i.i.d. r.v.'s, the well-known inequality  $E|\sum_i M_i|^p \leq 2 \sum_i E|M_i|^p$  for independent zero mean random variables  $M_i$  with  $E|M_i|^p < \infty$  and  $1 \leq p \leq 2$  (see [25]), the fact  $E|\eta_s|^p = C_p < \infty$  for some  $p \in (1, 2)$  and the Minkowski inequality. Using these facts, we obtain

$$\begin{aligned} E|U_2|^p &\leq 2C_p \sum_s \left( \sum_{h=1}^q \frac{1}{n} \sum_{t=1}^n |\psi(t-s) \psi'(t+h-s)| \right)^p \\ &\leq C \left( \sum_{h=1}^q \frac{1}{n} \sum_{t=1}^n \left( \sum_s |\psi(t-s) \psi'(t+h-s)|^p \right)^{1/p} \right)^p \end{aligned}$$



$$\begin{aligned}
&\leq C \left( \sum_{h=1}^q \left( \sum_{s=0}^{\infty} s_+^{p(d-1)} (h+s)^{p(d'-1)} \right)^{1/p} \right)^p \\
&\leq C \left( \sum_{h=1}^q \left( h^{p(d+d'-2)+1} \right)^{1/p} \right)^p \\
&\leq C \left( \sum_{h=1}^q h^{(d+d'-2)+(1/p)} \right)^p \leq C q^{p((1/p)+d+d'-1)}
\end{aligned}$$

and therefore  $E^{1/p} |U_2|^p = O(q^{d+d'+(1/p)-1}) = o(q^{d+d'})$ , as  $p > 1$ , proving (A.27) for  $i = 2$ .

Next, consider  $U_3$ . Using the fact that  $\sum_{s=1}^{\infty} s^{d-1} (t+s)^{d'-1} \leq C t^{d+d'-1}$  ( $t \geq 0$ ), we obtain

$$\begin{aligned}
EU_3^2 &\leq 2 \sum_{s \neq s'} \left( \sum_{h=1}^q \left( 1 - \frac{h}{q+1} \right) \frac{1}{n} \sum_{t=1}^{n-h} \psi(t-s) \psi'(t+h-s') \right)^2 \\
&\leq \frac{C}{n^2} \sum_{t,t'=1}^n \sum_{h,h'=1}^q \sum_{s,s'} |\psi(t-s) \psi'(t+h-s') \psi(t'-s) \psi'(t'+h'-s')| \\
&\leq \frac{C}{n^2} \sum_{t,t'=1}^n \sum_{h,h'=1}^q |t-t'|_+^{2d-1} |t-t'+h-h'|_+^{2d'-1} \\
&\leq \frac{Cq}{n} \sum_{|t| \leq n} \sum_{|h| \leq q} |t|_+^{2d-1} |t+h|_+^{2d'-1} \leq C(J_1 + J_2),
\end{aligned} \tag{A.28}$$

where

$$\begin{aligned}
J_1 &= (q/n) \sum_{|t| \leq 2q} |t|_+^{2d-1} \sum_{|h| \leq 3q} |h|_+^{2d'-1} \leq C(q/n) q^{2(d+d')} = o(q^{2(d+d')}), \\
J_2 &= (q^2/n) \sum_{2q < |t| \leq n} |t|_+^{2d+2d'-2} \leq C(q^2/n) \begin{cases} n^{2d+2d'-1}, & d+d' > 1/2 \\ q^{2d+2d'-1}, & d+d' < 1/2 \\ \log(n/q), & d+d' = 1/2 \end{cases}
\end{aligned}$$

and so  $J_2 = o(q^{2(d+d')})$  as  $q, n, n/q \rightarrow \infty$  in all three cases (in the last case  $d+d' = 1/2$  this follows from the fact that  $x \rightarrow 0$  entails  $x \log(1/x) \rightarrow 0$ ).

This proves the first relation of (A.24) for  $d > 0, d' > 0$ .

Consider now the case  $d = d' = 0$  we want to prove that

$$S'_q \rightarrow_p c,$$

where

$$\begin{aligned}
c &= \lim_{n \rightarrow \infty} n^{-1} E \left( \sum_{t=1}^n X(t) \right) \left( \sum_{s=1}^n X'(s) \right) \\
&= \tilde{\rho}_\xi \lim_{n \rightarrow \infty} n^{-1} \sum_u \sum_{t=1}^n \psi(t-u) \sum_{s=1}^n \psi'(s-u) \\
&= \tilde{\rho}_\xi \alpha \alpha'.
\end{aligned} \tag{A.29}$$

By writing  $S'_q = ES'_q + (S'_q - ES'_q)$ , the convergence  $ES'_q \rightarrow c$  follows similarly as in (A.29) above. Relation  $S'_q - ES'_q = o_p(1)$  can be shown similarly to (A.27), i.e., by splitting  $S'_q - ES'_q$  into “diagonal” and “off-diagonal” parts in the quadratic form in noise variables. Consider the “diagonal” part  $U_2$  in (A.25). Then

$$E|U_2|^p \leq 2C_p \sum_s \left( \sum_{h=1}^q \frac{1}{n} \sum_{t=1}^n |\psi(t-s) \psi'(t+h-s)| \right)^p \leq C(W_1 + W_2),$$

where

$$\begin{aligned} W_1 &= \sum_{s=1}^n \left( \sum_{h=1}^q \frac{1}{n} \sum_{t=1}^n |\psi(t-s)\psi'(t+h-s)| \right)^p \\ &\leq Cn^{-p} \sum_{s=1}^n \left( \sum_{t=1}^{\infty} |\psi(t-s)| \right)^p = O(n^{1-p}) = o(1) \end{aligned}$$

since  $p > 1$ , while

$$\begin{aligned} W_2 &= \sum_{s=0}^{\infty} \left( \sum_{h=1}^q \frac{1}{n} \sum_{t=1}^n |\psi(t+s)\psi'(t+h+s)| \right)^p \\ &\leq Cn^{-p} \sum_{s=0}^{\infty} \left( \sum_{t=1}^n |\psi(t+s)| \right)^p \\ &\leq C \left( n^{-1} \sum_{t=1}^n \left( \sum_{s=0}^{\infty} |\psi(t+s)|^p \right)^{1/p} \right)^p = o(1) \end{aligned}$$

where we used the Minkowski inequality and the dominated convergence theorem to get  $o(1)$ , in view of the fact that  $\sum_{s=0}^{\infty} |\psi(t+s)|^p$  is bounded in  $t$  and tends to zero as  $t \rightarrow \infty$ .

Consider the “off-diagonal” term  $U_3$  in (A.26). Noting that, for fixed  $h, h'$ , the sum in (A.28) over all  $t, t', s, s' \in \mathbb{Z}$  is bounded by a constant independent of  $h, h'$ , we get  $EU_3^2 \leq C(q/n)^2 = o(1)$ .

This proves the first relation of (A.24) for  $d > 0, d' > 0$  and  $d = d' = 0$ .

Let us prove the second relation in (A.24). It follows from

$$\sum_{|h| \leq q} E|\hat{\gamma}(h) - \tilde{\gamma}(h)| = o(q^{d+d'}). \quad (\text{A.30})$$

Using definitions of  $\hat{\gamma}(h)$ ,  $\tilde{\gamma}(h)$ , the Cauchy–Schwartz inequality and (2.5), for  $h \geq 0$  one obtains

$$\begin{aligned} (E|\hat{\gamma}(h) - \tilde{\gamma}(h)|)^2 &\leq E(\bar{X})^2 E\left(\frac{1}{n} \sum_{t=1}^{n-h} X'(t+h)\right)^2 + E(\bar{X}')^2 E\left(\frac{1}{n} \sum_{t=1}^{n-h} X(t)\right)^2 + E(\bar{X})^2 E(\bar{X}')^2 \\ &\leq Cn^{2d+2d'-2} \end{aligned} \quad (\text{A.31})$$

and so (A.30) reduces to  $Cqn^{d+d'-1} = o(q^{d+d'})$  which is a consequence of  $d + d' < 1$  and  $q/n \rightarrow 0$ .

This concludes the proof of (A.24) for  $d > 0, d' > 0$  and  $d = d' = 0$ . The cases  $d > 0 = d'$  and  $d = 0 < d'$  can be treated in a similar way. Proposition 3.1 is proved.  $\square$

### A.3. Derivation of the adaptive bandwidth

The aim of this section is to derive the adaptive bandwidth formula used in our simulations, viz.

$$\hat{q} = 0.3|\hat{l}|^{1/2} \begin{cases} n^{1/(3+4\hat{d})}, & \text{if } \hat{d} < 1/4, \\ n^{1/2-\hat{d}}, & \text{if } \hat{d} > 1/4, \end{cases} \quad (\text{A.32})$$

where  $\hat{d} = (\hat{d}_1 + \hat{d}_2)/2$  is an estimator of the (common) long memory parameter  $d$ ,

$$\hat{l} = \int_0^\pi \left( \frac{\hat{g}_1(x)}{\hat{g}_1(0)} - \frac{\hat{g}_2(x)}{\hat{g}_2(0)} \right) \frac{dx}{x^{2\hat{d}} \sin^2(x/2)}, \quad (\text{A.33})$$

and where  $\hat{g}_i$  is an estimator of the short memory part  $g_i(x) = f_i(x)/|x|^{2d_i}$  of the spectral density  $f_i$  of  $X_i$ ,  $i = 1, 2$ . In this paper,  $\hat{g}_i$  is the spectral density of the best AR approximation of  $g_i$  which is computed following the two step procedure in [7]. Namely, we first estimate  $d_i$  and then we fit an AR process to  $(1 - L)^{\hat{d}_i} X_i$  using the BIC criterion.

From [1, Theorem 2.1] under similar assumptions on  $X_i$  as in Section 2 we have the following expansion of  $S_{ii,q}$ : for  $0 < d_i < 1/4$ ,

$$q^{-2d_i} S_{ii,q} = c_{ii} + (q/n)^{1/2} g_i(0)(Z_{ni} + o_p(1)) + q^{-1-2d_i} g_i(0)(B_i + o_p(1)), \quad (\text{A.34})$$

where  $Z_{ni} \rightarrow_{\text{law}} Z_i \sim N(0, v(d_i))$ ,

$$v(d_i) = 8\pi \int_0^\infty \left( \frac{\sin(x/2)}{x/2} \right)^4 x^{-4d_i} dx,$$

$$B_i = \int_0^\infty \left( \frac{g_i(x)}{g_i(0) \sin^2(x/2)} \mathbf{1}_{\{0 < x < \pi\}} - \frac{1}{(x/2)^2} \right) \frac{dx}{x^{2d_i}}$$

and where  $g_i(x) = f_i(x)|x|^{2d_i}$  is the short memory component of the spectral density  $f_i$  of  $X_i$ , which is assumed to be continuous at  $x = 0$  and  $g_i(x) = g_i(0) + O(x^2)$ ,  $x \rightarrow 0$ ,  $g_i(0) > 0$ . Note that the long-run variance  $c_{ii}$  is related to  $g_i(0)$  by

$$c_{ii} = g_i(0)p(d_i), \quad (\text{A.35})$$

where  $p(d) = 2\Gamma(1-2d) \sin(\pi d)/d(1+2d)$  depends only on  $d$ .

From the form of statistic  $T_n$  it is clear that  $q$  must be chosen so that the ratio  $c_{11}/c_{22}$  is well estimated by  $S_{11,q}/S_{22,q}$ . From (A.34), assuming  $d_1 = d_2 = d$  as under the null hypothesis, we obtain

$$\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 = (q/n)^{1/2} \frac{Z_{n1} - Z_{n2}}{p(d)} (1 + o_p(1)) + q^{-1-2d} \frac{B_1 - B_2}{p(d)} (1 + o_p(1)). \quad (\text{A.36})$$

Therefore as  $n, q, q/n \rightarrow \infty$ ,

$$E \left( \frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 \right)^2 \sim \frac{1}{p(d)^2} ((q/n)E(Z_1 - Z_2)^2 + q^{-2(1+2d)}I^2), \quad (\text{A.37})$$

since for  $d_1 = d_2 = d$ , we have  $B_1 - B_2 = I$ , where

$$I = \int_0^\pi \left( \frac{g_1(x)}{g_1(0)} - \frac{g_2(x)}{g_2(0)} \right) \frac{dx}{x^{2d} \sin^2(x/2)},$$

c.f. (A.33). Minimizing the right-hand side of (A.37) with respect to  $q$ , we obtain

$$q = K_1(d)|I|^{2/(3+4d)}n^{1/(3+4d)}, \quad (\text{A.38})$$

where  $K_1(d)$  depends on  $d$  and  $E(Z_1 - Z_2)^2$ . Numerical computation of the function  $K_1(d)$  reveals that it is well approximated by the constant value 0.3 on the interval  $(0, 1/4)$  except for the case when  $d$  is close to  $1/4$  and then  $K_1(d)$  diverges but then also the approximations in (A.34) and (A.37) are less accurate. Therefore we choose to replace  $K_1(d)$  in (A.38) by 0.3 on the whole interval  $d \in (0, 1/4)$  in order that the test is not too conservative. For similar reasons, we replace the exponent of  $|I|$  in (A.38) by  $1/2$ , since otherwise the test turns out to be too conservative for small values of  $d$  when  $(X_1)$  and  $(X_2)$  have very different short memory parts (or high values of  $|I|$ ). The result of these replacements is  $q = 0.3|I|^{1/2}n^{1/(3+4d)}$ , c.f. (A.32).

Next, let us turn to the case  $1/4 < d < 1/2$ . From [1, Theorem 2.1], we obtain that for  $1/4 < d_i < 1/2$

$$q^{-2d_i}S_{ii,q}^\circ = c_{ii} + (q/n)^{1-2d_i}g_i(0)(\tilde{Z}_{ni} + o_p(1)) + q^{-1-2d_i}g_i(0)(B_i + o_p(1)), \quad (\text{A.39})$$

where  $\tilde{Z}_{ni} \rightarrow_{\text{law}} \tilde{Z}_i$  and  $\tilde{Z}_i$  is a (non-Gaussian) r.v. whose distribution depends only on  $d_i$ . From Proposition 2.2 (2.8) we have that

$$q^{-2d_i}(S_{ii,q} - S_{ii,q}^\circ) = -2(q/n)^{1-2d_i}g_i(0)(Y_{ni}^2 + o_p(1)), \quad (\text{A.40})$$

where

$$Y_{ni} = g_i(0)^{-1/2}n^{1/2-d_i}\tilde{X}_i \rightarrow_{\text{law}} Y_i \sim N\left(0, \frac{c_{ii}}{g_i(0)}\right) = N(0, p(d_i)), \quad (\text{A.41})$$

see (A.35). Combining (A.39)–(A.41) and using the facts that  $E\tilde{Z}_i = 0$ ,  $i = 1, 2$  and  $EY_1^2 = EY_2^2$ , similarly as in (A.36) and (A.37) we obtain

$$\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 = (q/n)^{1-2d} \frac{\tilde{Z}_{n1} - 2Y_{n1}^2 - \tilde{Z}_{n2} + 2Y_{n2}^2}{p(d)} (1 + o_p(1)) + q^{-1-2d} \frac{B_1 - B_2}{p(d)} (1 + o_p(1))$$

and

$$E \left( \frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 \right)^2 \sim \frac{1}{p(d)^2} ((q/n)^{2(1-2d)}EJ^2(d) + q^{-2(1+2d)}I^2), \quad (\text{A.42})$$

where  $J(d) = \tilde{Z}_1 - 2Y_1^2 - \tilde{Z}_2 + 2Y_2^2$  has a distribution depending on  $d$  alone. Minimization of the right-hand side of (A.42) with respect to  $q$  leads to

$$q = K_2(d)|I|^{1/2}n^{1/2-d}, \quad (\text{A.43})$$

where  $K_2(d)$  is a function of  $d$ . In this case, we also choose  $K_2(d) = 0.3$  for similar reasons as in the case  $d < 1/4$  above. This completes our derivation of the bandwidth formula (A.32).

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